

MAS 117

Introduction to Algebra

In-term Test

Sample paper solutions

1 (20 marks)

(a) What is meant by a *counterexample* to the statement

“For every natural number n , the number $n^2 + n + 5$ is a prime number.”

A *counterexample* to this statement is a natural number n such that $n^2 + n + 5$ is not a prime number.

(b) Find a counterexample to this statement.

The number $n = 4$ is a counterexample; for $4^2 + 4 + 5 = 25 = 5 \times 5$ is not a prime number.

2 (20 marks) Prove by induction that

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n \times (n+1)} = \frac{n}{n+1}$$

for any positive integer n .

Starting the induction: when $n = 1$, the left-hand side is $1/(1 \times 2)$ and the right-hand side is $1/2$, so they are equal.

Inductive step: suppose that the result is true for a value n . We have to prove that the result holds with $n + 1$ replacing n . The left-hand side, with $n + 1$ replacing n , is

$$\begin{aligned} \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n \times (n+1)} + \frac{1}{(n+1)(n+2)} &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n(n+2) + 1}{(n+1)(n+2)} \\ &= \frac{(n+1)^2}{(n+1)(n+2)} \\ &= \frac{n+1}{n+2} \end{aligned}$$

which is the right-hand side of the given equation with $n + 1$ replacing n . So the inductive step is complete.

By induction the result is true for all $n \geq 1$.

3 (20 marks) True or false? Give reasons.

(a) The operation of vector product satisfies the commutative law.

False. For $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, but $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$.

(b) The operation of addition on the natural numbers satisfies the inverse law.

False. For the identity element for addition is zero, and there is no natural number b such that $1 + b = 0$.

(c) The operation of scalar product satisfies the associative law.

False. For $\mathbf{u} \cdot \mathbf{v}$ is a scalar (a real number), so $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$ is not even defined.

4 (20 marks) What does it mean to say that a relation R on a set A is (i) *reflexive*, (ii) *symmetric*, (iii) *transitive*?

R is

- *reflexive* if $(a, a) \in R$ for all $a \in A$;
- *symmetric* if $(a, b) \in R$ implies $(b, a) \in R$;
- *transitive* if $(a, b), (b, c) \in R$ imply $(a, c) \in R$.

Which of these three properties does the relation “less than” on the set $\mathbb{Z}$ enjoy?

- Not reflexive since it is not the case that $1 < 1$.
- Not symmetric since $1 < 2$ but it is not the case that $2 < 1$.
- Transitive, since if $a < b$ and $b < c$, then $a < c$.

5 (20 marks) Use Euclid's Algorithm to find the greatest common divisor d of 69 and 144, and to express d in the form $69x + 144y$, where x and y are integers.

$$\begin{aligned}144 &= 2 \cdot 69 + 6, \\69 &= 11 \cdot 6 + 3, \\6 &= 2 \cdot 3.\end{aligned}$$

So the greatest common divisor of 69 and 144 is 3.

$$\begin{aligned}3 &= 69 - 11 \cdot 6 \\&= 69 - 11(144 - 2 \cdot 69) \\&= 23 \cdot 69 - 11 \cdot 144\end{aligned}$$

so $x = 23, y = -11$.